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GENERALIZED CROSSED PRODUCTS APPLIED TO MAXIMAL ORDERS, BRAUER GROUPS AND RELATED EXACT SEQUENCES

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0. Introduction

If G is any group and R is a graded ring of type $G, R = \bigoplus_{\sigma\in G} R_\sigma$, then R is said to be strongly graded by G if we have: $R_\sigma R_\tau = R_{\sigma\tau}$ for every $\sigma, \tau \in G$. In case G is a finite group we will refer to a strongly graded ring of type G as a generalized crossed product, a terminology stemming from T. Kanzaki [13]. These rings or the slightly more general Clifford systems have been studied extensively by E. Dade in [6,7], C. Năstăcescu, F. Van Oystaeyen in [17,18] and F. Van Oystaeyen in [26]. The results in this paper circle around the following main idea: if A is a certain maximal order over a commutative ring C and A contains a commutative extension S of C such that G acts as a group of C-automorphisms of S and such that S^G (the fixed ring of G) equals C, then A is a generalized crossed product over S with respect to G, i.e. $A = \bigoplus_{\sigma\in G} S_\sigma$, $S_e = S$ and $S_\sigma S_\tau = S_{\sigma\tau}$ for all $\sigma, \tau \in G$. The situations we actually consider are: A is a maximal Krull order over a Dedekind domain; A is a relative Azumaya algebra in the sense of [27] or in particular a reflexive Azumaya algebra in the sense of Yuan [29]; A is a common Azumaya algebra. Certain extra conditions have to be imposed on S and these have the effect that S becomes a 'relative' or a 'weak' Galois extension of C. In the first section we introduce some machinery as well as new results of quite general applicability concerning the (interrelated) topics: generalized crossed products, relative strongly graded rings for finite groups and graded Galois extensions. In the second section

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we extend some of the main results of [13] and [26] in two directions: first we relax the Galois-type conditions on the commutative subring S of A, secondly we allow A to be a maximal order (hence not necessarily an Azumaya algebra) over a Dedekind domain. The main tool here is a trace map associated to the gradation. Section 3 deals with th cohomological interpretation of certain results leading to the observation that for Azumaya algebras the generalized crossed product results are actually equivalent to the exact sequence of S. Chase, A. Rosenberg, whereas in earlier papers only one implication was obvious, cf. [13], [26]. This result is just a Galois-cohomological translation of a general result in Amitsur cohomology, Theorem 3.11. Finally we present the relative version of the Chase-Rosenberg-sequence and indicate how the reflexive case yields some interesting exact sequences in (relative) Galois cohomology over Krull domains, related to results of D.S. Rim, [21].

Section 3 is not really complete and maybe not completely well-balanced. Indeed, whereas we treat the absolute case rather extensively in Amitsur cohomology, the less-known relative theory is only hinted at in terms of Galois cohomology. The combination of Amitsur cohomology and relative theory will be the object of another paper; secondly, the exact sequences in relative Galois cohomology described at the end of the paper may indeed by viewed as an exercise if the reader is well aware of the absolute case and is willing to take for granted some results of [27]. For the general theory of graded rings we refer to [18].

1. Some basic results

1.1. New facts on strongly graded rings

If R is strongly graded by G, then it follows that each R_σ, $\sigma \in G$, is an invertible R_e-bimodule, where e is the unit element of G. The graded structure of R yields a group-homomorphism $\Phi: G \to \mathrm{Pic}(R_e)$, where $\mathrm{Pic}(R_e)$ is the Picard group of R_e consisting of the R_e-bimodule isomorphism classes of invertible R_e-bimodules; Φ is given by $\Phi(\sigma) = [R_\sigma]$, $\sigma \in G$. There exists a canonical morphism $\pi: \mathrm{Pic}(R_e) \to \mathrm{Aut}(Z(R_e))$, mapping the class of an invertible R_e-bimodule P to the automorphism $\pi(P)$ which is defined by $\pi(P)(c) = c'$, the unique element of $Z(R_e)$ such that $c'P = Pc$ elementwise. The composition of π and Φ defines an action of G on $Z(R_e)$. With this definition the action of $\sigma \in G$ on $Z(R_e)$ is then given by $\sigma(c)t = tc$ for all $c \in Z(R_e)$, $t \in R_\sigma$. For each $\sigma \in G$ we have that $R_\sigma R_{\sigma^{-1}} = R_e$ and hence we may fix for each σ a decomposition: $1 = \sum_i u_\sigma^{(i)} v_{\sigma^{-1}}^{(i)}$. If $c \in Z(R_e)$, then the action of σ on c may alternatively be described by $\sigma(c) = \sum_i u_\sigma^{(i)} c v_{\sigma^{-1}}^{(i)}$. The ring fixed in $Z(R_e)$ under the action of G will be denoted by R_0. Clearly R_0 is contained in the center of R because $R = \bigoplus_{\sigma \in G} R_\sigma$ and an element of R_0 commutes with each R_σ. We will consider R as an R_0-algebra.

1.1.1. Definition. The Φ-trace on $Z(R_e)$ is defined by $\mathrm{tr}_\Phi(c)=\sum_{\sigma\in G}\sigma(c)$.

In [26] F. Van Oystaeyen proved Maschke's theorem for Clifford systems of finite groups. In the following proposition we generalize the result (but for strongly graded rings) by weakening the condition imposed in [26] to a mild trace-condition and secondly we refine the result by showing that here is no dependence on the chosen decomposition of 1.

1.1.2. Proposition. *Let R be strongly graded by a finite group G and suppose that there exists an element of trace one, i.e. $\mathrm{tr}_\Phi(c)=1$ for some $c\in Z(R_e)$. Let $\varphi: M\to N$ be an R_e-linear map between left R-modules M and N. There exists a canonical left R-linear map $\tilde{\varphi}: M\to N$, only depending on φ and c. If $f: N\to T$ is a left R-linear map for some other left R-module T, then $f\tilde{\varphi}=f(\varphi)^{\sim}$.*

Proof. For $m\in M$, define

$$\tilde{\varphi}(m)=\sum_{\sigma\in G}\sum_i u_\sigma^{(i)}c\varphi(v_{\sigma^{-1}}^{(i)}m).$$

The proof that $\tilde{\varphi}$ is left R-linear is identical to the proof of the similar statement in the generalized Maschke's theorem as given in [26] but taking into the account that $1=\sum_{\sigma\in G}\sigma(c)$. Let us establish that the definition of $\tilde{\varphi}$ is independent of the choice of the decompositions: $1=\sum_i u_\sigma^{(i)}v_{\sigma^{-1}}^{(i)}$ for each $\sigma\in G$. Therefore we consider another set of decompositions: $1=\sum_j u_\sigma'^{(j)}v_{\sigma^{-1}}'^{(j)}$.

A straightforward calculation now yields, for $m\in M$:

$$\begin{aligned}\tilde{\varphi}(m)&=\sum_{\sigma\in G}\sum_i u_\sigma^{(i)}c\varphi(v_{\sigma^{-1}}^{(i)}m)\\&=\sum_{\sigma\in G}\sum_i u_\sigma^{(i)}c\varphi\left(v_{\sigma^{-1}}^{(i)}\sum_j u_\sigma'^{(j)}v_{\sigma^{-1}}'^{(j)}m\right)\\&=\sum_{\sigma\in G}\sum_i\sum_j u_\sigma^{(i)}cv_{\sigma^{-1}}^{(i)}u_\sigma'^{(j)}\varphi(v_{\sigma^{-1}}'^{(j)}m)\\&=\sum_{\sigma\in G}\sum_i\sum_j u_\sigma^{(i)}v_{\sigma^{-1}}^{(i)}u_\sigma'^{(j)}c\varphi(v_{\sigma^{-1}}'^{(j)}m)\\&=\sum_{\sigma\in G}\sum_j u_\sigma'^{(j)}c\varphi(v_{\sigma^{-1}}'^{(j)}m)=\tilde{\varphi}'(m).\end{aligned}$$

The final assertions in the proposition are easily verified. Note that exactly the presence of c makes $\varphi=\tilde{\varphi}$ if φ is R-linear.

1.1.3. Corollary. *In the situation of the proposition, we have:*

(a) (E. Nauwelaerts) *If R_0 is left hereditary, then R is left hereditary.*

(b) *If R_e is left regular, then R is left regular.*

(c) *If R_e is separable over R_0, then R is separable over R_0.*

Proof. (a) Basically the lifting of a left R_e-linear splitting map in an exact sequence of left R-modules to an R-linear splitting map.

(b) Left as an exercise.

(c) Similar to the proof of a similar result in [26].

Note that the condition: $|G|^{-1}\in R_0$, implies the condition: tr_Φ is surjective.

For application in the crossed product theory *we now restrict attention to the case where R_e is commutative*, i.e. we have $R_e = Z(R_e)$. Furthermore we extend tr_Φ to R in a rather trivial way, $\mathrm{tr}_\Phi(r) = \mathrm{tr}_\Phi(r_e)$ if $r = r_e + \cdots + r_\sigma$ is the decomposition of r into homogeneous components. We also define a bilinear R_0-form τ_Φ by putting $\tau_\Phi(s,t) = \mathrm{tr}_\Phi(st)$. In order to prove that τ_Φ is a symmetric associative bilinear form on R we need the following key-lemma.

1.1.4. Lemma. *Let R be strongly graded by a finite group G such that R_e is commutative. For arbitrary $s\in R_\sigma$, $t\in R_{\sigma^{-1}}$ we have $st = \sigma(ts)$.*

Proof. Let p_0 be a maximal ideal of R_0. The localization R_{p_0} of R at p_0 is strongly graded by G, with $(R_{p_0})_e = (R_e)_{p_0}$. Since taking the fixed ring for the action of G commutes with localization at prime ideals in the invariant ring we may identify $(R_0)_{p_0}$ and $(R_{p_0})_0 = R_0'$. Let $\bar{s},\bar{t}$ be the images of s,t respectively in R_{p_0}. We first prove that $\bar{s}\bar{t} = \sigma(\bar{t}\bar{s})(=\overline{\sigma(ts)})$. Write $R_e' = (R_{p_0})_e$, $p_0' = R_0' p_0$. Consider the set of maximal ideals $\{P_1',\dots,P_m'\}$ of R_e' lying over p_0'. If we show that G acts transitively on this set, then m is finite. Suppose G does not act transitively and suppose that Q_1' and Q_2' are maximal ideals of R_e' such that $Q_2'\notin\{\sigma(Q_1')\mid\sigma\in G\}$. Put $q_1' = \bigcap\{\sigma(Q_1'),\sigma\in G\}$. By the Chinese remainder theorem there exists an element $a'\in R_e'$ such that $a'\equiv 0 \pmod{Q_2'}$, $a'\equiv 1 \pmod{q_1'}$. Then $N(a') = \prod_{\sigma\in G}\sigma(a')$ also has the property that $N(a')\equiv 0 \pmod{Q_2'}$, $N(a')\equiv 1 \pmod{q_1'}$. Since $N(a')\in R_0'$ the first relation implies $N(a')\in p_0'$ whereas it follows from the second that $N(a')\notin p_0'$, a contradiction.

Consequently, R_e' is a semilocal ring and $\mathrm{Pic}(R_e') = 1$. It follows from this that $R' = \bigoplus_{\sigma\in G} R_e' u_\sigma'$ where u_σ' is a unit of R' (since $1\in R_e' u_\sigma' R_{\sigma^{-1}}' = R_e' u_\sigma' u_{\sigma^{-1}}'$). Obviously we may assume that $u_e' = 1$. Now $\bar{s} = x' u_\sigma'$, $\bar{t} = y'(u_\sigma')^{-1}$ and $\bar{s}\bar{t} = x'\sigma(y')$ while $\bar{t}\bar{s} = y'\sigma^{-1}(x')$. Consequently, $st - \sigma(ts)$ maps to zero in R_{p_0} for every maximal ideal p_0 of R_0, hence $st - \sigma(ts) = 0$ follows.

1.1.5. Corollary. *In the situation of the lemma, if $1 = \sum_i u_\sigma^{(i)} v_{\sigma^{-1}}^{(i)}$, then $1 = \sum_i v_{\sigma^{-1}}^{(i)} u_\sigma^{(i)}$.*

Proof. We have $\sigma(ts) = \sum_i u_\sigma^{(i)} t s v_{\sigma^{-1}}^{(i)}$, for $t\in R_{\sigma^{-1}}$, $s\in R_\sigma$. But then we may commute $u_\sigma^{(i)} t\in R_e$ with s and obtain:

$$\begin{aligned}\sigma(ts) &= \sum_i s\sigma^{-1}(u_\sigma^{(i)} t) v_{\sigma^{-1}}^{(i)} \\ &= \sum_i s v_{\sigma^{-1}}^{(i)}(u_\sigma^{(i)} t) = \sum_i s(v_{\sigma^{-1}}^{(i)} u_\sigma^{(i)})t \\ &= \sum_i st\sigma(v_{\sigma^{-1}}^{(i)} u_\sigma^{(i)}) = st\sigma\Big(\sum_i v_{\sigma^{-1}}^{(i)} u_\sigma^{(i)}\Big).\end{aligned}$$

By the lemma (and since the equalities hold for all $s\in R_\sigma$, $t\in R_{\sigma^{-1}}$) we have $1=\sigma(\sum_i v_{\sigma^{-1}}^{(i)}u_\sigma^{(i)})$ and the corollary follows. □

1.1.6. Remark. The corollary yields a generalization of the fact that a left inverse for $v\in R$ is also a right inverse (note that R is a P.I. ring since it is finite over a commutative ring). The authors were puzzled by the fact that it seems to be unavoidable to use the fact that R_e is semilocal in proving this. In the general strongly graded situation, putting $a=\sigma(\sum_i v_{\sigma^{-1}}^{(i)}u_\sigma^{(i)})$, one may easily calculate that $a^2=1$ and that $a=\sigma(a)=\sum_i v_{\sigma^{-1}}^{(i)}u_\sigma^{(i)}$. To conclude from this that $a=1$ seems to be impossible unless one uses the lemma.

1.1.7. Theorem. *If R is strongly graded by a finite group G such that R_e is commutative, then τ_Φ defines a symmetric associative R_0-bilinear form on R.*

Proof. Associativity and R_0-bilinearity are evident from the definition. In order to establish the symmetry of τ_Φ it will suffice to check this property for homogeneous elements, say $s\in R_\sigma$, $t\in R_\tau$. If $\tau\neq\sigma^{-1}$ then, $\tau_\Phi(s,t)=\tau_\Phi(t,s)=0$. If $\tau=\sigma^{-1}$, then we have

$$\tau_\Phi(s,t)=\mathrm{tr}_\Phi(st)=\mathrm{tr}_\Phi(\sigma(ts))=\mathrm{tr}_\Phi(ts)=\tau(t,s). \qquad \square$$

The techniques we developed in this section will be used in crossed product theory further on; let us recall here some definitions of generalized crossed products, referring to [13] [17] for more detail.

To an arbitrary group G and any ring A such that there is given a group morphism $\Phi: G\to\mathrm{Pic}(A)$, $\sigma\to[P_\sigma]$ we may associate strongly graded rings $A\langle f,\Phi,G\rangle$ (cf. [18, Proposition I.3.13]) depending on factor sets f with respect to Φ. Such a factor set is just a family of A-bimodule isomorphisms $f_{\sigma,\tau}: P_\sigma\otimes P_\tau\to P_{\sigma\tau}$ satisfying the usual associativity and unitary conditions expressed by the commutativity of the following diagrams:

$$\begin{array}{ccc}
P_\sigma\otimes P_\tau\otimes P_\theta & \xrightarrow{1\otimes f_{\tau,\theta}} & P_\sigma\otimes P_{\tau\theta} \\
\Big\downarrow{\scriptstyle f_{\sigma,\tau}\otimes 1} & & \Big\downarrow{\scriptstyle f_{\sigma,\tau\theta}} \\
P_{\sigma\tau}\otimes P_\theta & \xrightarrow[f_{\sigma\tau,\theta}]{} & P_{\sigma\tau,\theta}
\end{array}$$

$$\begin{array}{ccc}
P_\gamma\otimes P_e & \xrightarrow[i\otimes\alpha]{} & P_\gamma\otimes A \\
{\scriptstyle f_{\gamma,e}}\searrow & & \swarrow\cong \\
& P_\gamma &
\end{array}$$

$$\begin{array}{ccc}
P_e\otimes P_\gamma & \xrightarrow[\alpha\otimes i]{} & A\otimes P_\gamma \\
{\scriptstyle f_{e,\gamma}}\searrow & & \swarrow\cong \\
& P_\gamma &
\end{array}$$

where $\sigma, \tau, \theta, \gamma \in G$ and $\alpha : P_e \to A$ is a given A-bimodule isomorphism. The ring $A\langle f, \Phi, G\rangle$ is just $\bigoplus_{\sigma \in G} P_\sigma$ with multiplication defined by

$$xy = f_{\sigma,\tau}(x \otimes y), \quad \text{if } x \in P_\sigma,\ y \in P_\tau.$$

Conversely, every strongly graded ring R over G is obtained in this way by putting $A = R_e$ and $P_\sigma = R_\sigma$ for all $\sigma \in G$. From Theorem I.3.16 of [18] we recall that two factor sets f and g associated to Φ differ by an element $q \in Z^2(G, U(Z(A)))$; $g = qf$. The isomorphism classes of graded A-algebras then correspond bijectively to elements of $H^2(G, U(Z(A)))$.

1.1.8. Theorem. *If A is an Azumaya algebra with center C containing a Galois extension D of C as a maximal commutative subring, say* $\mathrm{Gal}(D/C) = G$, *then A is a strongly graded ring over G with $A_e = S$, i.e. there exists a group homomorphism* $\Phi : G \to \mathrm{Pic}(S)$, *and a factor set f associated to ϕ, such that $A \cong S\langle f, \Phi, G\rangle$.*

Proof. This is a consequence of a much more general theorem on relative Azumaya algebras, cf. [26], which we recall in Section 1.2, see also [13] for the absolute case. Note that in this case tr_Φ reduces to the usual Galois trace function in S.

1.2. Relative strongly graded rings

In this section R is a G-graded ring, G an arbitrary group. We consider an idempotent kernel functor κ on R_e-mod. For the general theory of kernel functors and localization we refer to [11] or [12]. We say that R is *relatively strongly graded with respect to κ*, or simply *κ-graded*, if $Q_\kappa(R_\sigma R_\tau) = R_{\sigma\tau}$ for all $\sigma, t \in G$, where Q_κ denotes the (left) localization functor in R_e-mod, associated to κ. If κ is trivial i.e. $\mathscr{L}(\kappa) = \{R_e\}$, then R is κ-graded if and only if it is strongly graded. Less trivial but equally interesting is the following example. Let R_e be a maximal order over a Krull ring and let κ_1 be the idempotent kernel functor $\bigwedge_{P \in X^1(R_e)} \kappa_P$ with idempotent filter $\mathscr{L}(\kappa_1) = \bigcap \{\mathscr{L}(P) \mid P \in X^1(R_e)\}$. In this situation κ_1 is a central kernel functor, however one can easily construct more complicated examples by using relative maximal orders in the sense of [16]. A κ_1-graded ring is now nothing but a divisorially graded ring as in [16], [17]. Recall that κ is said to be a *Noetherian kernel functor* if any ascending chain of left ideals I_n is such that $\bigcup_n I_n \subset \mathscr{L}(\kappa)$, then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $I_n \in \mathscr{L}(\kappa)$. In the sequel we always assume that κ is Noetherian; this condition is actually equivalent to the fact that Q_κ commutes with direct sums and it is clearly a very mild restriction. Let us verify that the condition is satisfied for κ_1 as introduced above. Let R_e be a maximal order over a Krull domain, and consider an ascending sequence of left ideals of R_e, $\{I_n \mid n \in \mathbb{N}\}$, such that $\bigcup I_n \in \mathscr{L}(\kappa_1)$. It is clear that we may assume all I_n to be ideals since κ_1 is central (i.e. $\mathscr{L}(\kappa_1)$ has a cofinal subset consisting of ideals). Since $Q_{\kappa_1}(I_n) = I_n^{**}$ holds here (cf. [16]) we may consider the following chain of divisorial ideals: $\cdots \subset Q_\kappa(I_n) \subset \cdots$. Since R_e satisfies the ascending chain condition on divisorial ideals it follows that all I_n are contained in $Q_{\kappa_1}(I_{n_0})$ for some $n_0 \in N$. If

$I_{n_0} \notin \mathscr{L}(\kappa_1)$, then also the union of the I_n cannot be in $\mathscr{L}(\kappa_1)$, because $\bigcup_n I_n \subset Q_{\kappa_1}(I_{n_0})$ and $I_{n_0} \in \mathscr{L}(\kappa_1)$ if and only if $Q_{\kappa_1}(I_{n_0}) \in \mathscr{L}(\kappa_1)$.

For simplicity's sake we assume that κ is a *Noetherian symmetric kernel functor*, i.e. we assume $\mathscr{L}(\kappa)$ has a cofinal subset consisting of ideals (actually in all applications κ will be a central kernel functor). With these hypotheses on κ we have:

1.2.1. Lemma. *R is κ-graded by G if and only if $Q_\kappa(RR_\tau) = R$ for all $\tau \in G$. In this case every graded left ideal of R has the property $Q_\kappa(L) = Q_\kappa(RL_e)$.*

Proof. If R is κ-graded, then each R_σ is κ-closed, $\sigma \in G$. Since κ is Noetherian, $R = \bigoplus_{\sigma \in G} R_\sigma$ is then also κ-closed (i.e. $Q_\kappa(R) = R$). Moreover, if $x \in R_\sigma$, then $R_{\sigma\tau^{-1}} R_{\tau\sigma^{-1}} x \subset RR_\tau$ and $R_{\sigma\tau^{-1}} R_{\tau\sigma^{-1}} \in \mathscr{L}(\kappa)$ yields that $Q_\kappa(RR_\tau) = R$. Conversely, suppose that $Q_\kappa(RR_\tau) = R$ for all $\tau \in G$. Consider $x \in R_e$. There is an $I \in \mathscr{L}(\kappa)$ such that $Ix \subset RR_\tau$, hence $Ix \subset (RR_\tau)_e$ or $Ix \subset R_{\tau^{-1}} R_\tau$, i.e. $x \in Q_\kappa(R_{\tau^{-1}} R_\tau)$. Consequently $R_e \subset Q_\kappa(R_{\tau^{-1}} R_\tau)$. Secondly, $R_\tau \subset RR_\tau$ yields $Q_\kappa(R_\tau) \subset R$. If $x \in Q_\kappa(R_\tau) - R_\tau$, then $Ix \subset R_\tau$ for some $I \in \mathscr{L}(\kappa)$; if we write $x = x_{\sigma_1} + \cdots + x_{\sigma_n}$, then $Ix_{\sigma_i} = 0$ whenever $\sigma_i \neq \tau$. Since R is κ-torsionfree, $x_{\sigma_i} = 0$ whenever $\sigma_i \neq \tau$ and therefore $Q_\kappa(R_\tau) = R_\tau$. In particular $Q_\kappa(R_e) = R_e$ and $R_e = Q_\kappa(R_{\tau^{-1}} R_\tau)$ for all $\tau \in G$. Now we first check the second statement in the lemma. If L is a graded left ideal and $x \in L_\tau$ for some $\tau \in G$, then $R_\tau R_{\tau^{-1}} x \subset RL_e$ and thus $L \subset Q_\kappa(RL_e)$ and $Q_\kappa(L) = Q_\kappa(RL_e)$ follows. Now for all $\gamma, \tau \in G$, $R_\gamma R_{\gamma^{-1}} R_{\gamma\tau} \subset R_\gamma R_\tau$ and $R_\gamma R_{\gamma^{-1}} \in \mathscr{L}(\kappa)$, hence $R_{\gamma\tau} = Q_\kappa(R_\gamma R_\tau)$ for all $\gamma, \tau \in G$.

1.2.2. Remark. For an arbitrary $I \in \mathscr{L}(\kappa)$, the R_e-module R/RI need not be κ-torsion. Of course if $I = R_{\tau^{-1}} R_\tau$ for some $\tau \in G$ the foregoing property does hold. We say that κ *is G-invariant* if for all $I \in \mathscr{L}(\kappa)$ and all $\sigma \in G$, $R_\sigma I R_{\sigma^{-1}} \in \mathscr{L}(\kappa)$. Clearly, whenever κ is central, then $\mathscr{L}(\kappa)$ has a cofinal subset consisting of ideals which are generated by their central part and therefore a central κ is automatically G-invariant.

1.2.3. Proposition. *Let R be κ-graded by G where κ is Noetherian symmetric and G-invariant. If M is a graded κ-closed left R-module, then $Q_\kappa(R \otimes_{R_e} M_e) \cong M$ as graded modules.*

Proof. Consider $0 \to K \to R \otimes_{R_e} M_e \xrightarrow{f} M$ where f is the canonical graded morphism of degree e. On the one hand $\kappa(M) = 0$ implies that $\kappa(R \otimes_{R_e} M_e) \subset K$. On the other hand, if $x \in K_\tau$ for some $\tau \in G$, write $x = \sum_i \lambda_\tau^{(i)} \otimes m_e^{(i)}$. From $f(x) = 0$ it follows that $R_\tau R_{\tau^{-1}} x \subset R_\tau \otimes_{R_e} R_{\tau^{-1}} \sum_i \lambda_\tau^{(i)} m_e^{(i)} = 0$, i.e. $x \in \kappa(R \otimes_{R_e} M_e)$ or $K = \kappa(R \otimes_{R_e} M_e)$. Consequently, the canonical map $Q_\kappa(R \otimes_{R_e} M_e) \underset{i}{\hookrightarrow} Q_\kappa(M) = M$ is monomorphic. But $Q_\kappa(R \otimes_{R_e} M_e)$ is κ-closed. Furthermore, if $m_\tau \in M_\tau$, then $R_\tau R_{\tau^{-1}} m_\tau \subset R_\tau M_e$, with $R_\tau R_{\tau^{-1}} \in \mathscr{L}(\kappa)$, and $m_\tau \in i(Q_\kappa(R \otimes_{R_e} M_e))$.

We established that i is onto. Finally, since $M_\tau / R_\tau M_e$ is κ-torsion, it follows that $Q_\kappa(R_\tau \otimes_{R_e} M_e) = M_\tau$ and i is graded of degree e, what proves the proposition.

1.2.4. Theorem. *Let R, κ and G be as before. Then the R_e-bimodule isomorphism class $[R_\sigma]$ of R_σ is in* $\mathrm{Pic}(R_e, \kappa)$ *for all $\sigma \in G$. Actually, there is a group homomorphism $G \to \mathrm{Pic}(R_e, \kappa)$, defined by $\sigma \to [R_\sigma]$.*

Proof. For the definition of the relative Picard group PicR_e, κ) and its basic properties we refer to [27].

Step 1. In the situation of the theorem, if M is graded left R-module, then $Q_\kappa(M)$ is graded too and the grading of $Q_\kappa(M)$ is compatible with that of M. To prove this claim, first note that $\kappa(M)$ is graded and so we may assume that M is a κ torsion free i.e. $M \subset Q_\kappa(M)$.

Write $M'_\tau = Q_\kappa(M_\tau) \subset Q_\kappa(M)$. For $\gamma \in G$ consider $R_\gamma x$, $x \in M'_\tau$. It is clear that $R_\gamma I R_{\gamma^{-1}} R_\gamma x \subset M_{\gamma\tau}$ if I has been chosen in $\mathscr{L}(\kappa)$ such that $Ix \subset M_\tau$. Since κ is G-invariant, $R_\gamma x \subset M'_{\gamma\tau}$ follows and so $M' = \bigoplus_{\gamma \in G} M'_\gamma$ is graded left R-module containing M.

Since κ is Noetherian it follows that Q_κ commutes with direct sums and therefore $M' = Q_\kappa(M)$, proving that $Q_\kappa(M)$ is graded with $Q_\kappa(M)_\gamma = Q_\kappa(M_\gamma)$ for all $\gamma \in G$.

Step 2. For $\tau \in G$, let $R(\tau)$ be the τ-shifted R-module obtained by putting $R(\tau)_\sigma = R_{\sigma\tau}$ for all $\sigma \in G$. Putting $M = R(\tau)$ in Proposition 1.2.3 we find

$$R(\tau) = Q_\kappa\Big(R \underset{R_e}{\otimes} R_\tau\Big) \quad \text{and} \quad R_{\sigma\tau} = Q_\kappa(R_\sigma \otimes R_\tau) \quad \text{for all } \sigma \in G.$$

In particular, for $\tau = \sigma^{-1}$ we obtain

$$R = Q_\kappa\Big(R_\sigma \underset{R_e}{\otimes} R_{\sigma^{-1}}\Big) = Q_\kappa(R_{\sigma^{-1}} \otimes R_\sigma).$$

In order to prove that each R_σ, $\sigma \in G$, is κ-invertible (in the sense of [27]), it is only necessary to establish that R_σ is κ-flat, in other words:

(1) If N is a κ-torsion left R_e-module, then also $R_\sigma \otimes_{R_e} N$ is κ-torsion.

(2) If $0 \to N \to M$ is exact in R_e-mod, then we obtain an exact sequence in R_e-mod:

$$0 \to K \to R_\sigma \underset{R_e}{\otimes} N \to R_\sigma \underset{R_e}{\otimes} M,$$

with K a κ-torsion left R_e-module.

To prove the first property, consider N as in (1) and put $M = Q_\kappa(R \otimes_{R_e} N)$. By the first step we know that M is graded and $M_e = Q_\kappa(N)$. Therefore we have

$$M = Q_\kappa\Big(R \underset{R_e}{\otimes} N\Big) = Q_\kappa(R \otimes M_e) = Q_\kappa(R \otimes Q_\kappa(N))$$

and so we may derive from $Q_\kappa(N) = 0$ that $Q_\kappa(R \otimes_{R_e} N) = 0$ and $R \otimes_{R_e} N$ is κ-torsion.

In order to prove (2) we may start from the exact sequence

$$0 \to K_1 \to R_{\sigma^{-1}} \underset{R_e}{\otimes} K \to R_{\sigma^{-1}} \underset{R_e}{\otimes} R_\sigma \otimes N \to R_{\sigma^{-1}} \underset{R_e}{\otimes} R_\sigma \otimes M$$

and by localizing at κ we obtain

$$0 \to Q_\kappa(\kappa_1) \to Q_\kappa\left(R_{\sigma^{-1}} \underset{R_e}{\otimes} K\right) \to Q_\kappa\left(R_{\sigma^{-1}} \underset{R_e}{\otimes} R_\sigma \underset{R_e}{\otimes} N\right) \cong N$$

$$\downarrow$$

$$Q_\kappa\left(R_{\sigma^{-1}} \underset{R_e}{\otimes} R_\sigma \underset{R_e}{\otimes} M\right) \cong M$$

Then, $Q_\kappa(R_{\sigma^{-1}} \otimes_{R_e} K) = 0$ yields that $R_{\sigma^{-1}} \otimes K$ is κ-torsion, hence by (1) $R_\sigma \otimes_{R_e} R_{\sigma^{-1}} \otimes \kappa$ is also κ-torsion and this finally leads to the fact that K is κ-torsion. □

The κ-graded rings usually appear in connection with the so called 'relative' theory involving relative Azumaya algebras, relative Galois extensions and crossed products, relative Picard- and class-groups, etc..., in the sense of [27]. Let us just recall some basic definitions and properties here.

Let C be any commutative ring, κ an idempotent kernel functor on C-mod with idempotent filter $\mathscr{L}(\kappa)$. Write $X(\kappa)$ for the set of prime ideals of C not in $\mathscr{L}(\kappa)$ and let $C(\kappa)$ be the set of ideals of C maximal with the property of not being in $\mathscr{L}(\kappa)$; then $C(\kappa) \subset X(\kappa)$.

Every localization C_p of C at $p \in X(\kappa)$ is the localization of R_q, for some $q \in C(\kappa)$, at the prime ideal $p_q = pR_q$ of R_q. We say that $M \in C$-mod is *κ-finitely generated* if there is a C-submodule M' of M which is finitely generated and such that M/M' is κ-torsion. $M \in R$-Mod is said to be κ-finitely presented if there exist a C-module M' and a C-linear $u: M' \to M$ such that M' is finitely presented and Ker(u), Coker(u) are both κ-torsion C-modules. The following characterizations of κ-Azumaya and κ-Galois extensions algebras stem from [28].

1.2.5. Lemma. *Let C be a commutative ring, κ an idempotent kernel function on* C-mod *such that C is κ-closed; let A be a κ-closed C-algebra which is κ-finitely presented C-module. Then A is a κ-Azumaya algebra over C if and only if A_p is an Azumaya algebra over C_p for every $p \in C(\kappa)$, (hence for every $p \in (\kappa)$ as well).*

1.2.6. Lemma. *Let D be a commutative κ-closed C-algebra which is a κ-progenerator and κ-separable (in the sense of* [28, p. 50]). *Assume there is a finite group G of C-automorphisms of D such that $C = D^G$, the fixed ring in D under the action of G. Then D is a κ-Galois extension of C if and only if D_p is a Galois extension with Galois group G of C_p, for every $p \in C(\kappa)$.*

It is clear that the relative objects we have introduced are defined locally just as the usual objects but only for a certain geometrically stable set of prime ideals of the groundring (there is an obvious link to sheaf theory for which we to [28]).

1.2.7. Theorem. *Let A be a κ-Azumaya algebra over C containing as a maximal commutative subring a κ-Galois extension D of C with Galois group G. Then A is κ-graded by G such that $A_e = D$.*

Proof. If $\sigma \in G$, define $A_\sigma = \{x \in A \mid xd = \sigma(d)x, \text{ for all } d \in D\}$.

The proof consists in checking that these A_σ, $\sigma \in G$, do indeed define a G-gradation of the desired type; we refer to [26] to for full detail.

The consequences of this theorem in terms of cohomology will be investigated in Section 3. If κ is trivial, i.e. $\mathscr{L}(\kappa) = \{c\}$, then Theorem 1.2.7 reduces to Theorem 1.1.8. Taking $\kappa = \kappa_1$ when C is a Krull domain (for the definition of κ_1 see the introduction to this section), the theorem gives a crossed product result for reflexive Azumaya algebras in the sense of [29], or also [19], which leads to a 'reflexive' version of the Chase–Rosenberg sequence, cf. Section 3. It is a useful consequence of Theorem 1.2.7 that κ-Azumaya algebras split by a κ-Galois extension may be studied by using relative techniques on one hand, and graded techniques on the other.

1.3. $\mathbb{Z}$-graded Galois extensions

We aim to apply some of the crossed product results to the situation where A contains a $\mathbb{Z}$-graded Galois extension D of C and to relate the $G = \mathrm{Gal}(D/C)$-gradation on A to the $\mathbb{Z}$-gradation. In this section we introduce graded Galois extensions and establish some basic properties which have not entered the literature before. It is acceptable that the existence of a graded structure on a commutative ring C has some influence on the structure of the Brauer group $\mathrm{Br}(C)$. The first job one then faces is to study the graded Azumaya algebras A over C and then to investigate the links between $\mathrm{Br}(C)$ and $\mathrm{Br}^g(C)$, the graded Brauer group in terms of graded Azumaya algebras. The latter also appears quite naturally in the study of Brauer groups of projective varieties. For all this we refer to [27]; for general graded ring theory, cf. [18].

Throughout this section C is a $\mathbb{Z}$-graded commutative ring and D is a graded C-algebra. The functor that forgets gradation is denoted by $\underline{} : C\text{-gr} \to C\text{-mod}$, where C-gr is the category of graded C-modules.

1.3.1. Lemma. *The idempotents of a $\mathbb{Z}$-graded commutative ring R are homogeneous of degree zero.*

Proof. If R is gr-local, i.e. R has a unique maximal graded ideal, then the result is easily verified, cf. e.g. [2]. In general, suppose e is and idempotent of R and let e_0 be the part of degree zero in the homogeneous decomposition of e. For every graded prime ideal P, $e-e_0$ maps to zero in $Q_p^g(R)=S^{-1}R$, where $S=(R-P)\cap h(R)$. Hence $(e-e_0)\in\kappa_P^g(R)$ for all graded prime ideals of R and therefore $e-e_0=0$. □

A graded Azumaya algebra over a graded ring C is nothing but an Azumaya algebra over C which happens to be graded, cf. [23]. A similar result holds for Galois extensions.

1.3.2. Theorem. *Let C be a $\mathbb{Z}$-graded ring and let D be a $\mathbb{Z}$-graded extension of C which is a Galois extension with Galois group G. All C-isomorphisms of D in G are homogeneous of degree zero.*

Proof. First we introduce some terminology, following F. De Meyer, E. Ingraham in [8]. Let $\Delta=\Delta(D:G)$ be the D-algebra defined by taking the free D-module generated by $\{u_\sigma \mid \sigma\in G\}$ and introducing a multiplication law by $(au_\sigma)(by_\tau)=a\sigma(b)u_{\sigma\tau}$ for all $\sigma,\tau\in G$. Let $\nabla=\nabla(D:G)$ be the D-algebra by taking the free D-module generated by $\{v_\sigma \mid \sigma\in G\}$ with multiplications law defined by $(av_\sigma)(bv_\tau)=ab\delta_{\sigma\tau}v_\sigma$. Both Δ and ∇ become graded D-algebras if we put $\deg u_\sigma=0$ and $\deg v_\sigma=0$. If D is a Galois extension of C, then it follows from [8, Proposition III.1.2], that there is an algebra isomorphism $f: D\otimes_C D\to\nabla(D:C)$ given by $f(a\otimes b)=\sum_{\sigma\in G} a\sigma(b)v_\sigma$.

Write $e_\sigma=f^{-1}(v_\sigma)$. Then e_σ is an idempotent, hence it has degree zero. It follows that f has degree zero. Now consider an element $b\in D$ which is homogeneous of degree $r\in\mathbb{Z}$. Let $p_\sigma:\nabla(D:G)\to D$ be the projection on the component Dv_σ. Then $p_\sigma\circ f(1\otimes b)=p_\sigma(\sum_{\sigma\in G}\sigma(b)\sigma_\sigma)=\sigma(b)$ actually has degree r and thus σ has degree zero. □

1.3.3. Corollary. *If D is a graded Galois extension of C with Galois group G, then the canonical isomorphisms:*

$$g:\Delta(D:G)\to\mathrm{Hom}_C(D,D),\qquad fD\otimes D\to\Delta(D:G)$$

are graded morphisms of degree zero.

Proof. That f has degree zero has been established above. To see that g has degree zero it suffices to not that each $\sigma\in G$ is a graded morphism of degree zero and that g is defined by $g(au_\sigma)(x)=a\sigma(x)$ for $a,x\in D$. □

As a further corollary we obtain the equivalence between the notion of graded Galois extensions and the notion of gr-Galois extensions which are defined completely in the framework of intrinsic graded theory.

1.3.4. Corollary. *Let D be a $\mathbb{Z}$-graded extensions of C and let G be a finite group of C-automorphisms of D. The following statements are equivalent.*

(1) *D is a graded Galois extension of C.*

(2)(i) $C=D^G$.

(ii) *For every* $\sigma\in$, $\sigma\in G$, σ *is graded morphism of degree zero.*

(iii) *For each gr-maximal ideal* Ω *of* D *and for each* σ *of* D *and for each* $\sigma\neq e$ *in* G, *there is an* $x\in D$ *such that* $\sigma(x)-x\notin\Omega$.

Proof. (1)$\Rightarrow$(2). Straightforward.

(2)$\Rightarrow$(1). If $\sigma\neq e$, consider the (graded) ideal of D generated by the elements $\sum_{j=1}^{n} x_j(y_j-\sigma(y_j))$ where x_j and y_j are homogeneous in D, say I. Then I cannot be in any gr-maximal ideal and so $I=D$. The sequel of the proof is now identical to the proof of the implication (5)$\Rightarrow$(2) in Proposition III.1.2 in [8], so we do not repeat it here.

1.3.5. Proposition (Imbedding property). *Let* D' *be a* $\mathbb{Z}$*-graded extension of* C *which is a finitely generated projective separable extension with no other idempotents but* 0 *and* 1. *Then there is a graded extension* D *of* D' *which is a graded Galois extension of* C *and such that* 0 *and* 1 *are still the only idempotents.*

Proof. An easy graded modification of the corresponding ungraded result, cf. [8, Theorem III.2.9].

1.3.6. Corollary. *If* D' *is as in Proposition* 1.3.5, *then every separable* C*-subalgebra* E *of* D' *is a graded subalgebra.*

Proof. Consider a Galois extension D as in Proposition 1.3.5 above. By the fundamental theorem of Galois theory (III.1.1. in [8]) there is a subgroup H of the Galois group of D over C such that $E=\{x\in D \mid \sigma(x)=x \text{ for all } \sigma\in H\}=D^H$. Since the automorphisms in H are graded morphisms of degree zero, and writing $x=x_{i_1}+\cdots+x_{i_n}$, with $x_{i_j}\in h(D)$, we obtain $x=\sigma(x)=\sigma(x_{i_1})+\cdots+\sigma(x_{i_n})$, and then the uniqueness of the homogeneous decomposition of x in D yields $x_{i_j}=\sigma(x_{i_j})$, i.e. $x_{i_j}\in D^H=E$ for all $j=1,\dots,n$. So we proved that E is graded. □

1.3.7. Remark. Some of the above results have analogies in the case of graded κ-Galois extensions. This is not surprising because, up to some coherence condition (κ-finite presentation), the relative theory is just a 'partial globalisation' of the local data obtained from the common theory. We did not go into details here in order to avoid unnecessary abstraction. Let us just mention the following useful corollary of Theorem 1.2.7 in the graded situation.

1.3.8. Proposition. *Let* C *be a* $\mathbb{Z}$*-graded commutative ring and let* κ^g *be a graded kernel functor in the sense of* [22], (*i.e.* $\mathscr{L}(\kappa^g)$ *has a cofinal subset consisting of graded ideals of* C). *Let* A *be a* $\mathbb{Z}$*-graded* κ^g*-Azumaya algebra containing as a maximal* (*graded*) *commutative subring a* $\mathbb{Z}$*-graded* κ^g*-Galois extension* D *of* C *wtih Galois group* G. *Then* A *is* κ^g*-graded by* G *such that* $A_e=D$ *and moreover the*

G-gradation is compatible with the $\mathbb{Z}$-gradation of A, i.e. each A_σ is a $\mathbb{Z}$-graded D-module or equivalently $[A_\sigma] \in \mathrm{Pic}^g(D, \kappa^g)$, where the latter subgroup of $\mathrm{Pic}(D, \kappa^g)$ is obtained by selecting out the classes of graded κ^g-invertible D-modules.

Proof. The definition of A_σ, $\sigma \in G$, in Theorem 1.2.7 obviously makes A_σ into a $\mathbb{Z}$-graded D-module, the other statements in the proposition follow from Theorem 1.2.7.

2. Application to maximal orders in central simple algebras

In the first part of this section we consider the following situation. Let K be a field and let $A = \sum_{\sigma\in G} Ku_\sigma$ be strongly graded by a finite group G such that $A_e = K$. Denote $K_0 = K^G$; $H = \{\sigma \in G \mid \sigma | K = 1_K\}$. Let R_0 be an integrally closed domain in K_0; write $u_\sigma u_\tau = f_{\sigma,\tau} u_{\sigma\tau}$ for all $\sigma, \tau \in G$ where $f_{\sigma,\tau}$ is a 2-cocycle, and we may assume $u_e = 1$. We also consider the H strongly graded ring $A' = \sum_{\sigma\in H} Ku_\sigma$.

We assume futher that $|H|^{-1} \in K_0$ and that tr_Φ is non-degenerate, where tr_Φ is defined as in Definition 1.1.1 and extended to a map on A as described before Lemma 1.1.4. These assumptions together with the generalized version of Maschke's result imply that A is K_0-separable. Since Φ is a rather obvious map, we write tr^0_{A/K_0} instead of tr_Φ in order to make some of the dependencies more explicit.

2.1. Proposition. *If $a \in A$ is integral over R_0, then $\mathrm{tr}^0_{A/K_0}(a) \in R_0$.*

Proof. First we deal with the case $H = G$, $A' = A$. In the left regular representation of A in terms of the K_0-basis $\{a_i u_\sigma \mid i = 1, \dots, n, \sigma \in G\}$, $\{a_1, \dots, a_n\}$ is some fixed K_0-basis for K, we easily calculate that the common trace T_{A/K_0} in this representation coincides with $\mathrm{tr}^0_{A/K}$ (i.e. if $a \in Ku_\sigma$, $\sigma \neq e$ then $T_{A/K_0}(a) = 0$ and for $a \in Ku_\sigma$, $\sigma = e$, $T_{A/K_0}(a) = na!$). Therefore the statement is clear in this case.

In general, write $B = K \otimes_{K_0} A = \sum_{\sigma\in G} (K \otimes_{K_0} K) u_\sigma$. Since K is a Galois extension of K_0 with group G/H it follows that $K \otimes_{K_0} K \simeq \sum_{\tau \in G/H} Ke_\tau$ where each $e_\tau r_\gamma = 0$ if $\gamma \neq \tau$. For $a \in K$ we have the identifications

$$a \otimes 1 = \sum_{\tau \in G/H} ae_\tau, \qquad 1 \otimes a = \sum_{\tau\in G/H} \tau(a) e_\tau.$$

Direct calculation yields

$$e_\mu B e_{\mu'} = \sum_{\sigma \in H\mu(\mu')^{-1}} \kappa u_\sigma$$

and we write $B_{\mu\mu'}$ for the latter. It is clear that $B_{\mu\mu} = A'$ and that each $B_{\mu\mu'}$ is an A'-bimodule.

Let $i : A \to K \otimes_{K_0} A$ be the canonical inclusion. Calculating the image $i(\lambda u_\sigma) \in h(A)$ in the representation of B as a matrix-algebra $(S_{\mu\mu'})$, μ and $\mu' \in G/H$, yields

$$e_\mu(1\otimes\lambda u_\sigma)e_{\mu'}=e_\mu\left(\sum_{\tau\in G/H}\tau(\lambda)e_\tau u_\sigma e_\mu\right),$$

and this is zero if $\sigma\notin H\mu(\mu')^{-1}$ and equal to $\mu(\lambda)u_\sigma$ otherwise. From the first part it follows that, for all $a\in A$,

$$\mathrm{tr}^0_{A/K_0}(a)=T_{A'/K_0}\left(\sum_{\mu\in G/H}e_{\mu i(a)e_\mu}\right).$$

Since A' is K_0-separable it is a semisimple Artinian algebra; we write $A'=\sum_{i=1}^m A_i'e_i$ for central idempotents e_i in $L_i=Z(A_i')$, $i=1,\dots,m$, and put $[A_i':L_i]=n_i^2$. Now we obtain

$$(*)\qquad T_{A'/K_0}\left(\sum_{\mu\in G/H}i(a)_{\mu\mu}\right)=\sum_{i=1}^m n_i\mathrm{tr}_{\mathrm{red}}(Be_i/K_0(i(a)e_i)$$

where $\mathrm{tr}_{\mathrm{red}}(Be_i/K_0)$ is the common reduced trace. Consequently, if a is integral over R_0, then so are all $i(a)e_i$ and then so is $(*)$, proving our claim.

2.2. Note. If A is a common crossed product, then tr^0_{A/K_0} is the common reduced trace.

Having explicited the trace tr^0_{A/K_0} in the case of a strongly graded algebra over a field, we are now ready to deal with maximal orders in separable algebras over fields. Consider a Dedekind ring R with field of fractions K and let Δ be strongly graded by the finite group G such that $\Delta_e=R$. We write A for $Q(\Delta)$ and hence we have that $A=\sum_{\sigma\in G}Ku_\sigma$. We assume that A satisfies the conditions imposed at the beginning of this section. Since R is a Dedekind ring, the fixed ring $R_0=R^G$ is again a Dedekind ring. We define the *inverse different* of R/R_0 with respect to tr^0_{K/K_0} as follows:

$$\mathcal{D}^{-1}_{R/R_0}=\{x\in K\mid \mathrm{tr}^0_{K/K_0}(xR)\subset R_0\}.$$

2.3. Proposition. *If $\Gamma\supset\Delta$ is a maximal R_0-order in A, then $\Delta\subset\Gamma\subset\mathcal{D}^{-1}_{R/R_0}\Delta$.*

Proof. Since the theorem is 'local' we may assume that R_0 is a discrete valuation ring. In this case R is semi-local and it follows that Δ has the form $\sum_{\sigma\in G}RU_\sigma$. From $\Delta\subset\Gamma$ it then follows that, for all $\gamma\in G$, $r\in R:\sum_r ra_\sigma u_\sigma u_{\gamma^{-1}}\in\Gamma$, hence $\mathrm{tr}^0_{A/K_0}(\sum_r ra_\sigma u_\sigma u_{\gamma^{-1}})\in R_0$, or $\mathrm{tr}_{K/K_0}(Ra_\gamma)\subset R_0$. This entails that for all $\gamma\in G$, $a_\gamma\in\mathcal{D}^{-1}_{R/R_0}$ or $\sum_{\sigma\in G}a_\sigma u_\sigma\in\mathcal{D}^{-1}_{R/R_0}\Delta$. □

In order to improve this result we calculate the discriminant of Δ/R_0 in terms of tr^0_{A/K_0}. Let $d^0(\Delta/R_0)$ be the ideal generated by

$$\{\det\mathrm{tr}^0(x_ix_j)\mid x_i\in\Delta, i,j=1,\dots,n\}\quad\text{where } n=\dim_{K_0}A=\mathrm{rk}_{R_0}\Delta.$$

2.4. Theorem. *With notations and hypotheses as above,* $d^0(\Delta/R_0)=d^0(R_e/R_0)^{|G|}$.

Proof. Again we may reduce the theorem to the case where R_0 is local i.e. a discrete valuation ring, and as in Proposition 2.3, Δ is of the form $\sum_{\sigma\in G} Ru_\sigma$. It will suffice to calculate $\det(\operatorname{tr} x_i x_j)$; $i,j=1,\dots,n$, for some R_0-basis $\{x_1,\dots,x_n\}$ of Δ. So consider $\{a_i u_\sigma \mid i,\sigma\in G\}$ as such a basis. Straightforward calculation now yields:

$$\chi=\det(\operatorname{tr}^0_{R/R_0}(a_i u_\sigma a_j u_\tau)_{ij})$$
$$=\sum_{\sigma\in G} det(tr^0_{R/R_0}(a_i\sigma(a_j)f_{\sigma,\tau}u_{\sigma\tau}))$$

The latter equality holds because in the first member only entries with $\tau=\sigma^{-1}$ contribute. Moreover, $\{\sigma(a_i)f_{\sigma,\sigma^{-1}} \mid i=1,\dots,r\}$ is an R_0-basis for R because $f_{\sigma,\sigma^{-1}}\in U(R)$, so we obtain further equalities:

$$\chi=\det(\operatorname{tr}^0_{R/R_0}(a_i\sigma(a_j)f_{\sigma,\sigma^{-1}})),$$

and thus $d^0(\Delta/R_0)=d^0(R_e/R_0)^{|G|}$. □

2.5. Corollary. *If* $\Gamma\supset\Delta$ *is a maximal* R_0*-order in* A, *then* $\Delta=\Gamma$ *if and only if* $d^0(\Gamma/R_0)=d^0(R/R_0)^{|G|}$.

Proof. In view of the foregoing result, the corollary may be proved in a classical way, cf. I. Reiner [20].

2.6. Remark. The obvious advantage of the foregoing theorem and the corollary is that we obtain from it a criterion for maximality of Δ in terms of R, R_0 and A alone. In the sequel of the section we are considering a kind of converse to this problem, i.e. we look for a criterion to decide whether certain maximal orders may be considered as strongly graded rings over certain commutative subrings.

An extension of Dedekind rings $R\supset R_0$ is said to be a pseudo-Galois extension if the following properties hold:

(1) If the fields of fractions of R_0, R, are K_0, K, resp., then K/K_0 is a Galois extension (with group G say).

(2) The extension R/R_0 is tamely ramified, or equivalently $\operatorname{tr}_{R/R_0}$ is surjective.

In the following proposition we use the techniques of the theory of Gr-Dedekind rings for which we refer to [24], [25].

2.7. Proposition. *If the extension of Dedekind domains* R/R_0 *ramifies in the primes* $P_1,\dots,P_n$ *with ramification indices* $e_1,\dots,e_n$, *resp. then we form*

$$\check{R}=\sum_{i\in\mathbb{Z}} Q_1^i\cdots Q_n^i X^i,\quad \text{and}\quad \check{R}_0=\sum_{i\in\mathbb{Z}} P_1^{[i/e_1]}P_2^{[i/e_2]}\cdots P_n^{[i/e_n]}X^i,$$

where $[\alpha]$ *denotes the smallest integer larger than* α, *and* $Q_i=\operatorname{rad}(RP_i)$.

Then $\check{R}/\check{R}_0$ *is a graded Galois extension in the sense of Section* 1.3.

Proof. From [25] it follows that both $\check{R}_0$ and $\check{R}$ are Gr-Dedekind rings. We may reduce the proposition to the gr-local case by localizing homogeneously at some P_i in R_0; so we just write P and Q in the sequel of the proof. It is easily verified that $\operatorname{rad}\check{R}$ equals $R\operatorname{rad}\check{R}_0$ and furthermore:

$$\check{S}=\check{R}/\operatorname{rad}\check{R}=\sum_{i\in\mathbb{Z}} R/QX^i,$$

$$\check{S}_0=\check{R}_0/\operatorname{rad}\check{R}_0=\sum_{i\in\mathbb{Z}} R_0/PX^{ie},$$

where e is the ramification index of P in Q. The fact that R/R_0 is tamely ramified implies, firstly that R/Q is a separable extension of R_0/P, secondly that $e\neq 0$, in R_0/P. Consequently, it follows that $\check{S}$ is separable over $\check{S}_0$ and thus also that $\check{R}$ is separable over $\check{R}_0$. Since Q is obviously G-invariant in R, the action of G on R can be extended in a natural way to an action of G on $\check{R}$ such that $(\check{R})^G=\check{R}_0$. That $\check{R}$ is a ($\mathbb{Z}$-graded) Galois extension of $\check{R}_0$ is thus obvious. □

The technique used in the above proof, i.e. taking generalized Rees rings in order to 'kill-off' the badly behaving ideals, has become a standard trick in the study of graded orders.

If $R_0\subset R$ is an extension of Dedekind rings and Δ is a maximal R_0-order in A, then Δ is said to *radicalize* R if $\Delta\supset R$ and for every maximal ideal P of R_0, $\operatorname{rad}\Delta_P=(\operatorname{rad}R_P)\Delta=\Delta(\operatorname{rad}R_P)$. This condition is clearly satisfied if Δ is an Azumaya algebra and R a Galois commutative subring, but also in the case where R_0 is local and A has a division ring for its residue algebra.

2.8. Proposition. *Let $R_0\subset R$ be an extension of Dedekind rings such that the field extension $K_0\subset K$, obtained by taking fields of fractions of R_0 and R respectively, is a Galois extension with group G. If Δ is a maximal R_0-order which radicalizes R, then there is a bijective correspondence between ideals of Δ and G-invariant ideals of R.*

Proof. We may assume that R_0 is a discrete valuation ring and in this case the G-invariant ideals of R are just the powers of $\operatorname{rad}(R)$ whereas on the other hand the ideals of Δ are the powers of $\operatorname{rad}(\Delta)$. Now first note that

$$(\operatorname{rad}(R))^n\Delta=(\operatorname{rad}(\Delta))^n=\Delta(\operatorname{rad}(R))^n.$$

We claim that R is a direct summand of Δ in R-mod. To see this it suffices to establish that Δ/R is a torsion free R-module (hence it will be projective!). Consider $a\in\Delta$, $s\in R$ such that $sa\in R$, i.e. $a\in\Delta\cap s^{-1}R$ and a is then certainly integral over R_0. However, R is the integral closure of R_0 in K, thus $a\in R$ and the claim follows. Write $\Delta=R\oplus\Delta'$ for some $\Delta'\in R$-mod. Then

$$(\operatorname{rad}(\Delta))^n\cap R=(\operatorname{rad}(R))^n\Delta\cap S$$

$$= ((\mathrm{rad}(R))^n \oplus (\mathrm{rad}(R))^n \Delta') \cap R$$

$$= (\mathrm{rad}(R))^n. \qquad \square$$

2.9. Proposition. *In the situation of proposition 2.8, if we assume moreover that R is a maximal commutative subring of Δ and that R is a pseudo-Galois extension of R_0 such that for all $P \in \Omega(R_0)$, $\Delta_P/\mathrm{rad}(\Delta_P)$ is a separable extension of R_0/P, then Δ is a generalized crossed product graded by G.*

Proof. Again, we may suppose that R_0 is local. Define $\check{R}_0$ and $\check{R}$ as in Proposition 2.7 and define $\check{\Delta} = \sum_{i\in\mathbb{Z}} (\mathrm{rad}(\Delta))^i X^i$. Since the central class group of $\check{\Delta}$ becomes trivial, it follows from [15] that $\check{\Delta}$ is a graded Azumaya algebra over $\check{R}_0$.

Proposition 2.7 entails that $\check{R}$ is a graded Galois extension of $\check{A}$. It is also clear that $\check{R}$ is a maximal commutative subring of $\check{\Delta}$. Theorem 1.1.8 yields that $\check{\Delta}$ is a generalized crossed product of $\check{R}$ and G with respect to some group morphism $\Phi : G \to \mathrm{Pic}(\check{R})$ and a $c \in H^2(G, U(R))$. Since the $\mathbb{Z}$-gradation and the G-gradation on $\check{\Delta}$ are compatible, (see Proposition 1.3.8 in the absolute case), it follows that $\Delta = \check{\Delta}_0$ is a generalized crossed product with respect to $R = (\check{\Delta})_0$ and G and the group morphism $\Phi : G \to \mathrm{Pic}(\check{R}) = \mathrm{Pic}^g(\check{R}) = \mathrm{Pic}((\check{R})_0) = \mathrm{Pic}(R)$. Note that $\mathrm{Pic}(\check{R}) = \mathrm{Pic}^g(\check{R})$ is a general fact for (commutative) $\mathbb{Z}$-graded Gr-Dedekind rings, whereas $\mathrm{Pic}^g(\check{R}) = \mathrm{Pic}((\check{R})_0)$ follows from the fact that $\check{R}$ is strongly $\mathbb{Z}$-graded. Note also that we used the fact that G is defined as a group of automorphisms, of $\check{R}$ defined in $\mathbb{Z}$-degree zero, i.e. on R over R_0. $\square$

3. Cohomological interpretation and some consequences

First we fix some terminology and we recall briefly the basic notions used in graded Amitsur cohomology, cf. [5] for full detail.

Throughout this section R is a $\mathbb{Z}$-graded commutative ring, S is a commutative graded R-algebra. We write $S^{(n)}$ for $S \otimes_R \cdots \otimes_R S$, the n-fold tensor product of S over R. If A and B are R-modules, then $\eta : A \otimes_R B \to B \otimes_R A$ is the switch map given by $\eta(a \otimes b) = b \otimes a$. For gven R-modules $M_1, \ldots, M_n$ we define ε_i, $i = 1, \ldots, n$, by putting

$$\varepsilon_i : M_1 \underset{R}{\otimes} \cdots \underset{R}{\otimes} M_n \to M_1 \underset{R}{\otimes} \cdots \underset{R}{\otimes} M_{i-1} \underset{R}{\otimes} S \underset{R}{\otimes} \cdots \underset{R}{\otimes} M_n,$$

$$(m_1 \otimes \cdots \otimes m_n) \to m_1 \otimes \cdots \otimes m_{i-1} \otimes 1 \otimes m_{i+1} \otimes \cdots \otimes m_n.$$

For any R-module M, put $M_1 = S \otimes_R M$, $M_2 = M \otimes_R S$, $M_{12} = M_{11} = S \otimes_R S \otimes_R M$, $M_{13} = M_{21} = S \otimes_R M \otimes_R S$, etc.

Any $S^{(2)}$-homomorphism $g : M_1 \to M_2$ induces homomorphisms $g_1 : M_{13} \to M_{23}$, $g_2 : M_{11} \to M_{23}$, $g_3 : M_{13} \to M_{23}$, in a natural way. From [2] we recall the following proposition.

3.1. Proposition. *For a $\mathbb{Z}$-graded commutative ring R, we have the following exact sequence of abelian groups:*

$$1 \to U_0(R) \to U(R) \xrightarrow{d} \mathrm{gr}(R) \to \mathrm{Pic}_g(R) \to \mathrm{Pic}^g(R) \to 1$$

Here $U_0(R)$ and $U(R)$ are the multiplicative groups of units in degree zero and in R respectively; $\mathrm{Pic}^g(R)$ and $\mathrm{Pic}_g(R)$ are graded Picard groups consisting of isomorphism classes, respectively isomorphism classes in degree zero, of graded R-progenerators of rank one (graded invertible modules); $\mathrm{gr}(R)$ is the group (multiplication induced by $\otimes_R$) of graded isomorphism classes in degree zero of graded R-progenerators of rank one which ae isomorphic to R as an ungraded R-module; the map d is defined by putting $d(u)$ equal to the graded R-module structure defined on R by putting $d(u)_m = R_m u$ (i.e. by giving u degree zero!), $m \in \mathbb{Z}$. If R is a reduced ring, $\mathrm{gr}(R)$ is described as follows.

3.2. Proposition. *Let R be a graded reduced commutative ring and consider $M \in \mathrm{gr}(R)$. Let $1 = e_1 + \cdots + e_n$ be the homogeneous decomposition of 1 in the gradation of M. Then the e_i are orthogonal idempotents in R, which are homogeneous in the R-gradation.*

Proof. Cf. [2].

Now consider the Amitsur complexes of the functors U and gr as defined in Proposition 3.1.

$$\begin{array}{ccccccc}
1 \longrightarrow & U(S) & \xrightarrow{\Delta_0} & U(S^{(2)}) & \xrightarrow{\Delta_1} & U(S^{(3)}) \\
& \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} \\
1 \longrightarrow & \mathrm{gr}(S) & \xrightarrow[D_0]{} & \mathrm{gr}(S^{(2)}) & \xrightarrow[D_1]{} & \mathrm{gr}(S^{(3)})
\end{array}$$

From this we derive the new complex:

$$1 \longrightarrow U(S) \times 1 \xrightarrow[\nabla_0]{} U(S^{(2)}) \times \mathrm{gr}(S) \xrightarrow[\nabla_1]{} U(S^{(3)}) \times \mathrm{gr}(S^{(2)})$$

by putting

$$\nabla_{n-1}(u, U) = \left(\Delta_{n-1} u, d_{n-1} u \underset{S^{(n-1)}}{\otimes} (D_{n-2}(U))\right).$$

We define the nth gr-cohomology group $H^2_{\mathrm{gr}}(S/R, U) = \mathrm{Ker}\, \nabla_n / \mathrm{Im}\, \nabla_{n-1}$.

We recall further from [2] the following general result.

3.3. Proposition. *With notations as before, we obtain the following exact cohomology sequence:*

$$1 \to H^0_{gr}(S/R, U) \to H^0(S/R, U) \to H^0(S/R, \mathrm{gr})$$
$$\to H^1_{gr}(S/R, U) \to H^1(S/R, U) \to H^1(S/R, \mathrm{gr})$$
$$\to H^2_{gr}(S/R, U) \to H^2(S/R, U).$$

In case S is a graded Galois extension of R, then we let $K^n(G, M)$ be the set of functions $G^n \to M$. From [14, Proposition V.1.6] we know that the mappings $\Phi_n : S^{(n+1)} \to K^n(G, S)$, defined by putting

$$\Phi_n(s_1 \otimes \cdots \otimes s_{n+1})\ (\sigma_1, \ldots, \sigma_n) = s_1 \sigma_1(s_2) \cdots \sigma_n(s_{n+1}),$$

yields an isomorphism of complexes. It is also evident that this is a graded isomorphism of degree zero (the elements of G have degree zero!) and consequently we can translate the graded Amitsur cohomology into Galois cohomology.

3.4. Corollary. *If S is a graded Galois extension with Galois group G, then*

$$H^n(S/R, \mathrm{gr}) = H^n(G, \mathrm{gr}(S)),$$
$$H^n(S/R, U_0) = H^n(G, U_0(S)) = H^n(G, U(S_0)),$$
$$H^n_{gr}(S/R, U) = H^n_{gr}(G, U(S)).$$

Furthermore we have the following long exact sequences:

$$1 \to H^0_{gr}(G, U(S)) \to U(R) \to H^0(G, \mathrm{gr}(S))$$
$$\to H^1_{gr}(G, U(S)) \to \mathrm{Pic}^g(R) \to H^0(G, \mathrm{Pic}^g(S))$$
$$\to H^2_{gr}(G, U(S)) \to \mathrm{Br}^g(S/R) \to H^1(G, \mathrm{Pic}(S)) \to H^3_{gr}(G, U(S))$$

and,

$$1 \to H^1(G, U_0(S)) \to \mathrm{Pic}_g(R) \to H^2(G, \mathrm{Pic}_g(S)) \to H^2(G, U_0(S))$$
$$\to \mathrm{Br}^g(S/R) \to H^1(G, \mathrm{Pic}_g(S)) \to H^3(G, U_0(S)).$$

3.5. Corollary. (a) *If* $\mathrm{Pic}_g R = 1$, *then* $H^1(G, U_0(S)) = 1$ (Hilbert's theorem 90).

(b) $\mathrm{Pic}_g S = 1$ *yields* $\mathrm{Br}^g(S/R) \cong H^2(G, U_0(S))$,

$\mathrm{Pic}^g S = 1$ *yields* $\mathrm{Br}^g(S/R) \cong H^2_{gr}(G, U(S))$.

3.6. Proposition. *Let R be a reduced graded commutative ring and let R be a graded Galois extension of R with Galois group G, then S is reduced and $H^1(G, \mathrm{gr}(S)) = 1$. The map $H^2_{gr}(G, U(S)) \to H^2(G, U(S))$, $[(u, U)] \to [u]$ is a monomorphism.*

Proof. That S is reduced is easily seen. A cocycle in $H^1(G, \mathrm{gr}(S))$ is in fact a group morphism from G to $\mathrm{gr}(S)$. Since G is finite and $\mathrm{gr}(S)$ is torsion free (follows from Proposition 3.2) it follows that $H^1(G, \mathrm{gr}(S)) = 1$. The final statement follows from Proposition 3.3 and Corollary 3.4. □

These techniques may be carried further so as to establish for gr-local rings R that $\mathrm{Br}^{g}(R) \subset \mathrm{Br}(R)$. We do not go into this here, but we prove a particular result for graded-Krull domains.

We need the following easy but technical lemma.

3.7. Lemma. *Let R be a $\mathbb{Z}$-graded Krull domain. Consider the kernel functor κ_1 introduced at the beginning of Section 1.2, and define $\kappa_{1,g}$ by taking for $\mathscr{L}(\kappa_{1,g})$ the idempotent filter generated by the graded ideals in $\mathscr{L}(\kappa_1)$. The following properties hold:*

(1) $\mathscr{L}(\kappa_{1,g}) = \bigcap \{\mathscr{L}(\kappa_p), P \in X^1(R)$ *and P is graded*$\}$.

(2) *$\kappa_{1,g}$ has finite type (and hence it is Noetherian).*

(3) *For every graded finitely generated R-module M which is κ_1-torsion free we have $Q_{\kappa_1}(M) = Q_{\kappa_{1,g}}(M) = Q^{g}_{\kappa_{1,g}}(M)$ where the latter is the graded localization at the graded kernel functor in the sense of* [22] *or* [18]. *Note that for every graded R-module M we always have that $Q_{\kappa_{1,g}}(M) = Q^{g}_{\kappa_{1,g}}(\mathrm{M})$.*

Proof. (1) If some prime ideal $P \in X'(R)$ contains a nonzero graded ideal, then $P = P_g$ and thus P is graded; the property follows immediately.

(2) Consider a graded ideal I in $\mathscr{L}(\kappa_{1,g})$ and suppose $S \subset \mathscr{L}(I)$ is such that the ascending chain $Ra_1, R_1 + Ra_2, \ldots, Ra_1 + \cdots + Ra_i$, $a_i \in S$; does not terminate. Then we obtain a chain of divisorial ideals $(Ra_1)^{**} \subset (Ra_1 + Ra_2)^{**} \subset \ldots$, and this chain terminates because R satisfies the ascending chain condition on divisorial ideals. Consequently, $(Ra_1 + \cdots + Ra_i)^{**} = R$ for some $i \in \mathbb{N}$. But since $Ra_1 + \cdots + Ra_i$ is finitely generated, $Q_{\kappa_1}(Ra_1 + \cdots + Ra_i) = (Ra_1 + \cdots + Ra_i)^{**} = R$, hence $Ra_1 + \cdots + Ra_i \in \mathscr{L}(\kappa_1)$ and then also $Ra_1 + \cdots + Ra_i \in \mathscr{L}(\kappa_{1,g})$. We established that every graded ideal in $\mathscr{L}(\kappa_{1,g})$ contains a finitely generated graded ideal in $\mathscr{L}(\kappa_{1,g})$ and this proves the property.

(3) Since $\kappa_{1,g}$ has finite type:

$$\varinjlim_{I \in \mathscr{L}(\kappa_{1,g})} \mathrm{Hom}_R(I, M) = \varinjlim_{I \in \mathscr{L}(\kappa_{1,g})} \mathrm{HOM}_R(I, M).$$

Indeed $\mathscr{L}(\kappa_{i,g})$ contains a cofinal set of finitely generated ideals and for such an ideal $\mathrm{Hom}_R(I, M)$ equals the graded $\mathrm{HOM}_R(I, M)$.

If M is a finitely generated graded module, then $M^* = \mathrm{Hom}_R(M, R) = \mathrm{HOM}_R(M, R)$ and M^{**} are both graded R-modules. Moreover, since $M^{**} = Q_{\kappa_1}(M)$ is graded, it is a direct consequence of graded localization results (cf. [22] or [18]) that $Q^{g}_{\kappa_{1,g}}(M) = Q_{\kappa_{1,g}}(M)$; whenever M is κ_1-torsion free. □

3.8. Proposition. *Let R be a $\mathbb{Z}$-graded Krull domain. The canonical morphism (forgetful!)* : $\mathrm{Br}^{g}(R) \xrightarrow{i} \mathrm{Br}(R)$ *is a monomorphism.*

Proof. Consider a graded Azumaya algebra over R representing an element α of $\mathrm{Br}^{g}(R)$ which maps to the trivial element of $\mathrm{Br}(R)$ under i, i.e. $A \cong \mathrm{End}_R(P)$ for some finitely generated projective R-module P. Write K^g for the graded field of fractions of R. It is clear that $K^g \otimes_R A$ is trivial in $\mathrm{Br}(K^g)$ and since the canonical map $\mathrm{Br}^g(K^g) \to \mathrm{Br}(K^g)$ is monomorphic (cf. [24]) it follows that there is a K^g-vector-space V of rank n, with homogeneous K^g-basis $\{v_1, \dots, v_n\}$ say, such that

$$K^g \underset{R}{\otimes} A \cong \mathrm{End}_{K^g}(V) = \mathrm{END}_{K^g}(V).$$

Put $F = \sum_{i=1}^n R v_i$. Clearly $E = AF$ is a graded finitely generated R-submodule of V containing a K^g-basis for V. Consequently $\mathrm{END}_R(E) = \mathrm{End}_R(E)$ is a gr-order in $\mathrm{End}_{K^g}(V)$. Since $A \subset \mathrm{End}_R(E) \subset \mathrm{End}(E^{**})$ and since A is a gr-maximal order (because it follows that $A = \mathrm{End}_R(M)$ for some graded reflexive R-module M. Morita equivalence fo R and A entails the existence of $N \in R$-mod such that $M \simeq P \otimes_R N$ (since M is an A-bimodule via $A = \mathrm{End}_R(M)$). Since M is reflexive, $\kappa_1(M) = 0$, and since P is faithfully projective $\kappa_1(N) = 0$ as well. The inclusion $0 \to N \to Q_{\kappa_1}(N)$ gives rise to

$$0 \to N \underset{R}{\otimes} P \to Q_{\kappa_1}(N) \underset{R}{\otimes} P \simeq Q_{\kappa_1}\left(N \underset{R}{\otimes} P\right) = Q_{\kappa_1}(M) = M.$$

Again by the faithful projectivity of P, it follows that $N = Q_{\kappa_1}(N)$ or N is a reflexive R-module. Since $\mathrm{End}_R(M) = \mathrm{End}_R(P) = A$, it is clear that N has rank one. Furthermore, the fact that $\mathrm{Cl}^g(R) = \mathrm{Cl}(R)$ entails that we may assume (up to R-isomorphism) that N is a $\mathbb{Z}$-graded R-module, i.e. $N \in \mathrm{Pic}^g(R)$. Hence there is a graded R-module N' such that $Q^g_{\kappa_{1,g}}(N \otimes_R N') \cong R$ as graded R-modules, i.e. in degree zero. Here we have used Lemma 3.7(3) in expressing the bidual as a graded localization, and we use this at various points in the following calculation. First:

$$Q_{\kappa_1}\left(M \underset{R}{\otimes} N'\right) = Q_{\kappa_{1,g}}\left(M \underset{R}{\otimes} N'\right) = Q^g_{\kappa_{1,g}}\left(M \underset{R}{\otimes} N'\right).$$

Secondly,

$$Q_{\kappa_1}\left(M \underset{R}{\otimes} N'\right) = Q_{\kappa_1}\left(P \underset{R}{\otimes} N \underset{R}{\otimes} N'\right) = P \underset{R}{\otimes} Q_{\kappa_1}(N \otimes N')$$

$$= P \underset{R}{\otimes} Q^g_{\kappa_{1,g}}(N \otimes N') \simeq P.$$

Since $Q^g_{\kappa_{1,g}}(M \otimes N') = Q_{\kappa_1}(M \otimes_R N')$ is in a canonical way a $\mathbb{Z}$-graded R-module we may transport this graded structure on P and thus α is trivial in $\mathrm{Br}^g(R)$. □

Now we investigate the relations between generalized crossed products and Amitsur- and Galois-cohomology. Since the formulation of the Amitsur-cohomology methods in the relative case require a lot of technicalities and some new concepts, we leave this for a forthcoming paper and restrict attention here to Galois cohomology in the relative setting. First we deal with the absolute case.

Let S be a commutative faithfully projective extension of R, an arbitrary commutative ring. We denote the nth Amitsur cohomology group of a functor F by $H^n(S(R,F))$ and simplify this to $H^n(F)$ if no confusion is possible. If $n<0$, then we put $H^n(F)=1$. The classical Chase-Harrison-Rosenberg sequence, cf. [4], may thus be written as

$$1 \longrightarrow H^{-2}(\mathrm{Pic}) \xrightarrow{\alpha_0} H^0(U) \xrightarrow{\beta_0} U(R) \xrightarrow{\gamma_0} 1 = H^{-1}(\mathrm{Pic})$$

$$\xrightarrow{\alpha_1} H^1(U) \xrightarrow{\beta_1} \mathrm{Pic}(R) \xrightarrow{\gamma_1} H^0(\mathrm{Pic}) \xrightarrow{\alpha_2} H^2(U)$$

$$\xrightarrow{\beta_2} \mathrm{Br}(5/R) \xrightarrow{\gamma_2} H((\mathrm{Pic}) \xrightarrow{\alpha_3} H^3(U).$$

For symmetry reasons we introduced the first row; exactness here is obvious since $H^0(U)=U(R)$ by the theorem of faithfully flat descent of elements, cf. [14]. We use terminology and notations of Knus, Ojanguren [14].

Let $I\in \mathrm{Pic}(S^{(2)})$ represent $[I]\in \mathrm{Ker}\,\alpha_3$. Then there exists an $S^{(3)}$-isomorphism $f: I_1 \otimes_{S^{(3)}} I_3 \to I_2$, (notations introduced at the beginning of this section). Then $f_4^{-1}f_2^{-1}f_3f_1$ is just multiplication by some cocycle $\omega\in U(S^{(4)})$ and $\alpha_3(I)$ is defined to be $[\omega]$. Since $[\omega]=1$ in $H^3(U)$ we may replace f by sf for some $s\in U(S^{(3)})$ such that $f_4^{-1}f_2^{-1}f_3f_1=1$.

Write I_S for the abelian group I considered as an S-module for the action $s\cdot i=(s\otimes 1)i$, (i.e. S acts on the first factor). Let $u\in U(S^{(3)})$ represent $[u]\in H^2(U)$. The composition g of abelian group homomorphisms

$$g: \left(S \underset{R}{\otimes} I_S\right) \underset{S^{(2)}}{\otimes} I = I_1 \underset{S^{(3)}}{\otimes} I_3 \xrightarrow{f} I_2 \xrightarrow{m} I_2 \xrightarrow{\eta} I_3 = I_S \underset{R}{\otimes} S,$$

turns out to be an $S^{(2)}$-module homomorphism. Furthermore

$$g_2^{-1}g_3g_1 = u_2^{-1}u_3u_1f_2^{-1}f_3f_1 = u_4f_2^{-1}f_3f_1$$

because

$$g_1=\eta_{11}f_1, \quad g_3=\eta_{14}f_4=\eta_{14}\eta_{11}f_3\eta_{11} \quad \text{and} \quad g_2=\eta_{14}\eta_{12}f_2.$$

Consider the $S^{(3)}$-linear map:

$$f_4=f_2^{-1}f_3f_1: \left(S \underset{R}{\otimes} S \underset{R}{\otimes} I_S\right) \underset{S^{(4)}}{\otimes} I_{14} \underset{S^{(4)}}{\otimes} I_{34} \to \left(S \underset{R}{\otimes} S \underset{R}{\otimes} I_S\right) \underset{S^{(4)}}{\otimes} I_{24}.$$

By conjugation f_4 defines an element of $\mathrm{End}_{S^{(3)}}(S\otimes_R S\otimes_R I_S)$. From this it follows that the element of $\mathrm{End}_{S^{(2)}}(S\otimes_R I_S)$ induced by g is a descent datum defining an Azumaya algebra $A=A(I,u,f)$ such that $A\otimes_R S=\mathrm{End}_S(I_S)$, i.e. $[A]\in \mathrm{Br}(S/R)$. Comparing this with the constructions in [14] we easily derive that

(1) $\quad [A(I,u,f)]\in \gamma_2^{-1}([I])$,

(2) $\quad [A(S^{(2)},u,1]=\beta_2(u)$.

Note that $A(I,u,f)$ contains S as a maximal commutative subalgebra.

From [14] we recall further:

(3) $A(S^{(2)}, 1, 1) = \mathrm{End}_R(S)$.

(4) If $s \in U(S^{(2)})$, $\Delta_2 s = s_1 s_2^{-1} s_3$, then

$$[A(S^{(2)}, u\Delta s, 1)] = [A(S^{(2)}, u, 1)].$$

(5) $[A(S^{(2)}, uv, 1)] = [A(S^{(2)}, u, 1)][A(S^{(2)}, v, 1)]$.

(6) The opposite algebra $A(S^{(2)}, u, 1)^0$ equals $A(S^{(2)}, u^{-1}, 1)$.

(7) $$[A(I, 1, f)][A(I', 1, f')] = \left[A\left(I \underset{S^{(2)}}{\otimes} I, 1, f \underset{S^{(2)}}{\otimes} f'\right)\right].$$

(8) $A(I, u, vf) = A(I, uv, f)$.

In all this only (7) requires a proof as a new result; (8) is easily verified. The proof of (7) is a modification of the proof given for (4) in [14], modulo some technicalities. We include the proof for completeness' sake.

3.9. *Proof of property* (7) *above.* We consider

$$A = A(I, 1, f), \qquad A' = A(I', 1, f'),$$

$$B = A\left(I \underset{S^{(2)}}{\otimes} I', 1, f \underset{S^{(2)}}{\otimes} f'\right), \qquad B' = A(S^{(2)}, 1, 1) \cong \mathrm{End}_R S.$$

From the theorem of faithfully flat descent of algebras we obtain

$$A \subset \mathrm{End}_S(I_S), \quad A' \subset \mathrm{End}_S(I'_S), \quad B \subset \mathrm{End}_S\left(\left(I \underset{S^{(2)}}{\otimes} I'\right)_S\right), \quad B' \subset \mathrm{End}_S(S_S^{(2)}).$$

Claim. An element $x \in \mathrm{End}_S((I \otimes_{S^{(2)}} I')_S \otimes_S S_S^{(2)})$ is in $B \otimes_R B'$ if and only if $f' \times (f')^{-1} \in A \otimes_R A'$. If we prove this claim, then property (7) follows. Now $A \otimes_R A'$ is obtained from the descent datum Φ_1 induced by $\eta_{12} f_2' \eta_{14} f_4 = \eta_{12} \eta_{14} f_3' f_4$:

$$\left(S \underset{R}{\otimes} \left(I_S \underset{S}{\otimes} I'_S\right)\right) \underset{S^{(2)}}{\otimes} I \underset{S^{(2)}}{\otimes} I' \to \left(I_S \underset{S}{\otimes} I'_S\right) \underset{R}{\otimes} S.$$

On the other hand $B \otimes_R B'$ is obtained from the descent datum Φ_2 induced by $\eta_{12} \eta_{14} f_4' f_4$:

$$S \underset{R}{\otimes} \left(\left(I \underset{S^{(2)}}{\otimes} I'\right)_S \underset{S}{\otimes} S_S^{(2)}\right) \underset{S^{(2)}}{\otimes} \left(I \underset{S^{(2)}}{\otimes} I'\right) \underset{S^{(2)}}{\otimes} S^{(2)} \to \left(\left(I \underset{S^{(2)}}{\otimes} I'\right)_S \underset{S}{\otimes} S_S^{(2)}\right) \underset{R}{\otimes} S.$$

Suppose $f' x (f')^{-1} \in A \otimes_R A'$, i.e. $\Phi_1(f' x (f')^{-1})_1 = (f' x (f')^{-1})_2$, or equivalently

$$\eta_{12} \eta_{14} f_3' f_4 f_1' x_1 (f_1')^{-1} (f_4)^{-1} (f_3')^{-1} \eta_{14} \eta_{12} = f_4' x_2 (f_4')^{-1}.$$

Using the fact that

$$\eta_{14} \eta_{12} f' = f_4' \eta_{14} \eta_{12} = \eta_{12} \eta_{14} (f_2')^{-1} f_3' f_4 f_1' x_1 (f_1')^{-1} (f_4)^{-1} (f_3')^{-1} f_2' \eta_{14} \eta_{12} = x_2$$

and observing that f_1' and f_4 do commute because they act on I' and I respectively, whereas moreover $f_4' = (f_2')^{-1} f_3' f_1'$ holds, we finally obtain

$$\eta_{12}\eta_{14} f_4' f_4 x_1 f_4 f_4' \eta_{14}\eta_{12} = x_2,$$

which states exactly that $x \in B \otimes_R B'$. □

3.10. Corollary. *From the observations above it follows that*

$$[A(I,u,f)][A(I',u',f')] = \left[A\left(I \underset{S^{(2)}}{\otimes} I', u \otimes u', f \otimes f'\right)\right]$$

and also that multiplying u or f by $\Delta_2 s$ for some $s \in (S^{(2)})$ does not affect $[A(I,u,f)]$. Let us also point out that every Azumaya algebra A over R which is split by S has to be equivalent to some $A(I,u,f)$ by the exactness of the Chase–Rosenberg sequence.

Proof. The first statements are clear from the foregoing. For the second statement consider such an Azumya algebra A and put $\gamma_2(A) = [I]$, $A' = A(I,1,f)$. Then $A' \in \gamma_2^{-1}([I])$, i.e. $\gamma_2([A \otimes_R (A')^{-1}]) = 1$ and thus $C = A \otimes_R (A')^{-1} \in \operatorname{Im} \beta_2$. The latter entails that C is equivalent to some $A(S^{(2)}, u, 1)$ and hence

$$[A] = [A(I,1,f)][A(S^{(2)},u,1)] = [A(I,u,f)].$$

So we have actually proved that $\operatorname{Br}(S/R) \cong \{A(I,u,f_I)$, where $I \in \operatorname{Ker}(\alpha_3)$, $u \in \operatorname{Coker}(\alpha_3)\}$ with multiplication defined by

$$[A(I,n,f_1)][A(I',u',f_{I'})] = \left[A\left(I \underset{S^{(2)}}{\otimes} I', \alpha_{I,I'} uu', f_{I \otimes_{S^{(2)}} I'}\right)\right].$$

3.11. Theorem. *Let S be a commutative faithfully projective extension of R. Then with notations as before:*

(1) $\operatorname{Br}(S/R) \cong \Delta(\operatorname{Ker}\alpha_3, \operatorname{Coker}\alpha_2, \alpha)$ *for some* $\alpha \in H^2(\operatorname{Ker}\alpha_3, \operatorname{Coker}\alpha_2)$,

(2) $\operatorname{Pic}(R) \simeq \Delta(\operatorname{Ker}\alpha_2, \operatorname{Coker}\alpha_1, \alpha)$ *for some* $\alpha \in H^2(\operatorname{Ker}\alpha_2, \operatorname{Coker}\alpha_1)$,

(3) $U(R) \simeq \Delta(\operatorname{Ker}\alpha_1, \operatorname{Coker}\alpha_0, \alpha)$ *for some* $\alpha \in H^2(\operatorname{Ker}\alpha_1, \operatorname{Coker}\alpha_0)$,

where $\Delta(\ )$ means crossed products with respect to the cocycle α.

Proof. (1) Fix a set of representatives $\mathcal{I} = \{I \mid [I] \in \operatorname{Ker}\alpha_3\}$ and $f_I : I_1 \otimes_{S^{(3)}} I_3 \to I_2$ such that $(f_I)_3 (f_I)_1 = (f_I)_2 (f_I)_4$. In the sequel of the proof we 'identify' $\mathcal{I}$ and $\operatorname{Ker}\alpha_3$.

For some cocycle $\alpha_{I,J}$ in $U(S^{(3)})$ we have $f_I \otimes f_J = \alpha_{I,J} f_{I \otimes J}$. Furthermore it follows from $(f_I \otimes f_J) \otimes f_K = f_I \otimes (f_J \otimes f_K)$ that $\alpha_{I,J}\alpha_{I\otimes J,K} = \alpha_{I,J\otimes K}\alpha_{J,K}$; or $\alpha \in H^2(\operatorname{Ker}\alpha_3, \operatorname{Coker}\alpha_2)$. Indeed, for any $s \in U(S^{(2)})$ the representatives for f_I, f_I or $f_{I\otimes J}$ may be modified in such a way that $\alpha_{I,J}$ is replaced by $(\Delta_2 s)\alpha_{I,J}$. Secondly, another choice of representatives f_I' leads to $f_I' = \beta_I f_I$ with β_I being a cocycle in $U(S^{(3)})$ and one easily verifies that the transformation formula for $\alpha_{I,J}$ is given by $\alpha_{I,J}' = \beta_I \beta_J \beta_{IJ}^{-1} \alpha_{I,J}$. This means that $[\alpha] = [\alpha']$ in $H^2(\operatorname{Ker}\alpha_3, \operatorname{Coker}\alpha_3)$.

Combining these remarks with Corollary 3.10, in particular the final lines of its proof, we see that we have proved (1).

(3) is equivalent to $U(R)=H^0(U)$, what follows from the exactness of the Chase–Rosenberg sequence.

(2) Note that $\operatorname{Coker}\alpha_1 = H^1(U)$. Let $I\in \operatorname{Pic}(S)$ represent $[I]\in \operatorname{Ker}\alpha_2$, and let $u\in U(S^{(2)})$ represent $[u]\in \operatorname{Coker}\alpha_1$. Then there exists an $f: I_1 \to I_2$ such that $f_3 f_1 = f_2$. It is thus obvious that uf is a descent datum defining an element $P(I,u,f)$ of $\operatorname{Pic}(R)$. The further deductions in this part follow the lines of proof as in (1); we leave this verification to the reader. □

It is now clear how to derive from the foregoing theorem the result of Theorem 1.1.8 in case S is a Galois extension of R, just by translating the descent theory developed above to the galois cohomological equivalent. So we have showed that Theorem 1.1.8 is actually equivalent to the Chase–Rosenberg (Galois-)sequence whereas earlier it had only been observed that latter sequence follows (only using elementary properties of the cohomology) directly from the generalized crossed product theorem 1.1.8. Since we have established before that the G-gradation on an Azumaya algebra containing a Galois extension S/R with $\operatorname{Gal}(S/R)=G$ as a maximal commutative subring is compatible with the $\mathbb{Z}$-gradation on A over R (if it exists), it follows that we may reformulate the foregoing theory for the graded Galois cohomology groups $H_g^n(G,\cdot)$; but we do not go into the details here.

For some final remarks we return to the relative case. As a consequence of Theorem 1.2.7 we obtain the following exact sequence (in the situation of 1.2.7):

$$1\to H^1(G,U(S))\to \operatorname{Pic}(R,\kappa)\to H^0(G,\operatorname{Pic}(S,\kappa))$$
$$\to H^2(G,U(S))\to \operatorname{Br}(S/R,\kappa)\to H^1(G,\operatorname{Pic}(S,\kappa))\to H^3(G,U(S))$$

To prove this fact one proceeds just as in the absolute case, taking care to replace Pic( ) by Pic($,\kappa$).

In [21] D.S. Rim proved the following result: if $S\supset R$ is an integral extension of an integrally closed Noetherian domain such that the extension of the field of fractions $L\supset K$ is a Galois extension with groups G, then the following is an exact sequence:

$$0\to C(S/R)\to H'(S/R)\to D(S)^G/i\,D(R)\to \operatorname{Cl}(S)^G/i\,\operatorname{Cl}(R)$$
$$\to H^2(S/R)\to \beta(S/R)\to H^1(G,\operatorname{Cl}(S))\to H^3(S/R)$$

where D stands for the divisor group and $C(S/R)$ is defined as the kernel of $i: D(R)\to D(S)$. Using the fact that $\beta(R)\hookrightarrow \operatorname{Br}(K)$ it is not so hard to derive a sequence related to this sequence from the relative sequence with respect to $\kappa=\kappa_1$ (note Pic($,\kappa_1$) = Cl( ), Br($,\kappa,$) = β( )) up to some easy modification of the first three terms in Rim's sequence. Actually, a slight extension of our approach yields that most of Rim's results may be stated for Krull domains R, i.e. not necessarily Noetherian integrally closed domains. Finally, since $\operatorname{Br}^g(R)\hookrightarrow \operatorname{Br}(R)$ for a graded

Krull domain and since $\beta^g(R)\hookrightarrow\beta(R)$ follows in a similar way, similar sequences may be derived from the graded cohomology. Some of these problems will be the topic of a forthcoming paper.

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